TEMPERATURE IN A WEDGE DUE TO A SOURCE OF VARIABLE SIZE AND STRENGTH

A. N. Reznikov and Yu. A. Novoselov

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A derivation is given of the temperature equation for a wedge with arbitrary included angle and boundary conditions of the second kind, on the surface of which there acts a strip source of variable dimensions and intensity. An example of the calculation of dimensionless temperature is examined, and its distribution over the boundary surfaces of the wedge is given.

When evaluating certain thermal processes in engineering and metallurgical practice, it is necessary to calculate the temperature in wedge-shaped bodies whose surfaces contain concentrated heat sources of variable intensity and frequently with dimensions which change with time. For example, problems of this kind arise in the calculation of tool temperatures in different types of milling.

We shall examine an infinite wedge, unrestricted in width, with included angle β (Fig. 1). We note, as may be shown by calculation, that a bounded wedge may be assumed unbounded if its length in the direction of the ρ-axis exceeds the length of the source $b(\tau)$ by a factor of 6–8. This proportion obtains as a rule in the bits of cutting tools.

We shall locate a heat source of variable intensity on the upper boundary of the wedge, which we shall designate the active boundary. The width of the contact area changes with time according to some law $b(\tau)$. It has been shown in a number of papers concerned with different areas of technical thermophysics [1, 2] that the distribution of heat source intensity is close to normal. We shall therefore conduct our examination for a normally distributed source, noting, however, that the method proposed below may be used also for solving problems with sources having other distributions.

We shall express the dependence of source strength on time and on the coordinate of the point of the contact surface $q(\tau, r)$ in the form of the product of two functions, the first depending only on τ, and the second on the polar radius r (Fig. 1). With a normal distribution, the second function has the form $\exp[-r^2/2\sigma^2]$, and therefore

$$q(\tau, r) = f(\tau)\exp[-r^2/2\sigma^2]. \tag{1}$$

In this expression $f(\tau)$ is the dimensional part of the intensity and its maximum value, while the exponential is a dimensionless multiplier. We shall assume the wedge boundaries to be adiabatic, and the problem to be a plane one.

On the active boundary of the wedge we shall choose an infinitely small increment dr of radius in the region of action of the source at a distance r from the coordinate origin. Since the wedge and its sources are not restricted in the direction perpendicular to the plane of the sketch, the chosen element may be taken as a line. Using the known expression [1] for an instantaneous linear source, acting in an infinite body,

$$\Theta = \frac{ql}{4\pi\lambda t}\exp\left[-\frac{R^2}{4at}\right],$$

we may write an expression for calculating the temperature at any point M (ρ, φ) of the infinite body due to the action of a strip source of strength $q(\tau, r)$, if the source heats the body during a time τ_0:

$$\Theta(\rho, \varphi, t) = \frac{1}{4\pi\lambda}\int_0^{\tau_0}\int_0^{b(\tau_0)}\frac{q(\tau, r)\,dr\,d\tau}{t-\tau}\times$$
$$\times\exp\left[-\frac{\rho^2+r^2-2\rho r\cos\mu}{4a(t-\tau)}\right]. \tag{2}$$

Here μ is the angle between the plane source and the radius vector drawn from the strip to the point M in question. Now, substituting into (2) the expression for the intensity (1), we obtain

$$\Theta(\rho, \varphi, t) = \frac{1}{4\pi\lambda}\int_0^{\tau_0}\int_0^{b(\tau_0)}\frac{f(\tau)\,dr\,d\tau}{t-\tau}\exp\left[-\frac{r^2}{2\sigma^2}-\right.$$
$$\left.-\frac{\rho^2+r^2-2\rho r\cos\mu}{4a(t-\tau)}\right]. \tag{3}$$

We shall introduce further dimensionless parameters, relating the linear and time quantities, respectively, to the upper limits of integration:

$$r/b(\tau_0) = \nu_i, \quad \rho/b(\tau_0) = \nu, \quad \tau/\tau_0 = \zeta_i, \quad t/\tau_0 = \zeta.$$

Then

$$\Theta(\rho, \varphi, t) = \frac{b(\tau_0)}{4\pi\lambda}\int_0^1\int_0^1\frac{\psi(\zeta_i)\,d\nu_i\,d\zeta_i}{\zeta-\zeta_i}\times$$
$$\times\exp\left[-\frac{\nu_i^2}{2\sigma^2/b^2(\tau_0)}-\frac{\nu^2+\nu_i^2-2\nu\nu_i\cos\mu}{[4a\tau_0/b^2(\tau_0)](\zeta-\zeta_i)}\right].$$

Designating additionally the groups

$$2\sigma^2/b^2(\tau_0) = \varepsilon, \quad 4a\tau_0/b^2(\tau_0) = \varkappa,$$

we may write the previous expression in the form

$$\Theta(\rho,\ \varphi,\ t) = \frac{b(\tau_0)}{4\pi\lambda} \int_0^1 \int_0^1 \frac{\psi(\zeta_i)\,d\nu_i\,d\zeta_i}{\zeta - \zeta_i} \times$$

$$\times \exp\left[-\frac{\nu_i^2}{\varepsilon} - \frac{\nu^2 + \nu_i^2 - 2\nu\nu_i\cos\mu}{\varkappa(\zeta - \zeta_i)}\right] =$$

$$= \frac{b(\tau_0)}{4\pi\lambda} \int_0^1 \frac{\psi(\zeta_i)\,d\zeta_i}{\zeta - \zeta_i} \exp\left[-\frac{\nu^2}{\varkappa(\zeta - \zeta_i)}\right] \times$$

$$\times \int_0^1 \exp\left[-\frac{\nu_i^2 - 2\nu\nu_i\cos\mu}{\varkappa(\zeta - \zeta_i)} - \frac{\nu_i^2}{\varepsilon}\right] d\nu_i. \qquad (4)$$

Taking the inner integral in (4), with the aid of the formulas given in [3], making the substitution $\varkappa/\varepsilon = \xi$, and carrying out a number of transformations, we obtain an equation for the temperature of a point of the infinite body with heating:

$$\Theta(\rho,\ \varphi,\ t) = \frac{1}{4\lambda}\sqrt{\frac{a\tau_0}{\pi}} \int_0^1 \frac{\psi(\zeta_i)\,d\zeta_i}{\sqrt{\zeta - \zeta_i}} \times$$

$$\times \frac{\exp\left[\dfrac{\nu^2}{\varkappa(\zeta - \zeta_i)}\left(\dfrac{\cos^2\mu}{1 + \xi(\zeta - \zeta_i)} - 1\right)\right]}{\sqrt{1 + \xi(\zeta - \zeta_i)}} \times$$

$$\times \left\{\operatorname{erf}\left[\frac{-\nu\cos\mu}{\sqrt{\varkappa(\zeta - \zeta_i)}}\,\frac{1}{\sqrt{1 + \xi(\zeta - \zeta_i)}} + \frac{\sqrt{1 + \xi(\zeta - \zeta_i)}}{\sqrt{\varkappa(\zeta - \zeta_i)}}\right] - \right.$$

$$\left. - \operatorname{erf}\left[\frac{-\nu\cos\mu}{\sqrt{\varkappa(\zeta - \zeta_i)}}\,\frac{1}{\sqrt{1 + \xi(\zeta - \zeta_i)}}\right]\right\}. \qquad (5)$$

Analysis of the quantities appearing in (5) shows that the expression $\sqrt{1 + \xi(\zeta - \zeta_i)}/\sqrt{\varkappa(\zeta - \zeta_i)}$ takes values of 2–3 and more. The second term, in square brackets, is many times less. Therefore the value of the first probability integral differs little from unity. We shall take it to be unity, which simplifies the expression considerably. The error introduced by this substitution even in the worst cases does not fall outside the range 1–2%.

Then

$$\Theta(\rho,\ \varphi,\ t) = \frac{1}{4\lambda}\sqrt{\frac{a\tau_0}{\pi}} \int_0^1 \frac{\psi(\zeta_i)\,d\zeta_i}{\sqrt{\zeta - \zeta_i}} \times$$

$$\times \frac{\exp\left[\dfrac{\nu^2}{\varkappa(\zeta - \zeta_i)}\left(\dfrac{\cos^2\mu}{1 + \xi(\zeta - \zeta_i)} - 1\right)\right]}{\sqrt{1 + \xi(\zeta - \zeta_i)}} \times$$

$$\times \operatorname{erfc}\left[-\frac{\nu\cos\mu}{\sqrt{\varkappa(\zeta - \zeta_i)}}\,\frac{1}{\sqrt{1 + \xi(\zeta - \zeta_i)}}\right]. \qquad (6)$$

In the temperature formulas obtained, the factor in front of the integral is the dimensional part, while the integral itself is a dimensionless functional coefficient. We shall designate it

$$\eta = \int_0^1 \operatorname{erfc}\left[-\frac{\nu\cos\mu}{\sqrt{\varkappa(\zeta - \zeta_i)}}\,\frac{1}{\sqrt{1 + \xi(\zeta - \zeta_i)}}\right] \times$$

$$\times \exp\left[\frac{\nu^2}{\varkappa(\zeta - \zeta_i)}\left(\frac{\cos^2\mu}{1 + \xi(\zeta - \zeta_i)} - 1\right)\right] \times$$

$$\times \frac{\psi(\zeta_i)}{\sqrt{1 + \xi(\zeta - \zeta_i)}}\,\frac{d\zeta_i}{\sqrt{\zeta - \zeta_i}} \qquad (7)$$

and express the temperature in terms of this functional coefficient:

$$\Theta(\rho,\ \varphi,\ t) = \frac{1}{4\lambda}\sqrt{\frac{a\tau_0}{\pi}}\,\eta. \qquad (8)$$

Formula (7) is valid for an unbounded body. To calculate the temperature in the wedge, it is necessary to perform a reflection, i.e., reduce the wedge to an infinite body. Using the method of approximate reflection of sources in wedges with any included angle, as previously described by the authors in [4], we obtain

$$\Theta(\rho,\ \varphi,\ t) = \frac{1}{4\lambda}\sqrt{\frac{a\tau_0}{\pi}}\left(2\sum_{n=1}^{k}\eta_n + \delta\eta_d\right) =$$

$$= \frac{1}{4\lambda}\sqrt{\frac{a\tau_0}{\pi}}\,T. \qquad (9)$$

In the last expression

$$\delta = 360°/\beta - 2k,$$

where the quantity k is taken to be an even integer when $|\delta| < 2$. The integers n = 1, 2, 3, ..., k are the number of reflected sources acting on the system. The location of these sources is determined by the angles

$$\varphi_n = (-1)^n\{n - [1 - (-1)^n]/2\}\beta,$$

and the quantities μ_n required for calculation of η_n from (7) are determined by the difference $\mu_n = \varphi - \varphi_n$. The angle describing the location of a fractional additional source [4], $\varphi_d = 180° + \beta$ and $\mu_d = \varphi - \varphi_d$ serve to calculate the value η_d from (7).

As an illustration, we shall calculate the dimensionless temperature in a wedge with angle $\beta = 72°$, and give a picture of the temperature distribution on the wedge surfaces at the end of the heating period ($\zeta = 1$).

Let the source intensity vary with time according to the law $\psi(\zeta_i) = \sin\zeta_i$. We note that this kind of variation of source intensity may occur in examining the cutting process when the cut thickness varies (for example, in milling). We assume $\varkappa = \xi = 1$, and then write

$$\eta = \int_0^1 \operatorname{erfc}\left(-\frac{\nu\cos\mu}{\sqrt{\zeta_i^2 - 3\zeta_i + 2}}\right) \times$$

$$\times \exp\left(\frac{\nu^2\cos^2\mu}{\zeta_i^2 - 3\zeta_i + 2} - \frac{\nu^2}{1 - \zeta_i}\right)\frac{\sin\zeta_i\,d\zeta_i}{\sqrt{\zeta_i^2 - 3\zeta_i + 2}}. \qquad (10)$$

We choose values of k and δ according to the formula $\delta = 360°/72° - 2k = 5 - 2k$ such that k does not exceed 2 in absolute value, k necessarily being even. Thus k can only be 2, and then $\delta = +1$. This means that the number of double integral sources is 2, the additional source being equal in power to the acting source ($\delta = 1$).

We now determine the angles describing the location of the sources taking part in the reflection (φ_1, φ_2, and φ_d):

$$\varphi_1 = (-1)^1 \{1 - [1 - (-1)^1]/2\} \cdot 72° = 0,$$

$$\varphi_2 = (-1)^2 \{2 - [1 - (-1)^2/2\} \cdot 72° = 144°,$$

$$\varphi_d = 180° + 72° = 252°.$$

We calculate the dimensionless temperature T at point A ($\nu = 1$; $\varphi = 0$). For this point we calculate the values $\mu_n = \varphi - \varphi_n$ and $\mu_d = \varphi - \varphi_d$:

$$\mu_1 = 0 - 0 = 0; \quad \mu_2 = 0 - 144° = -144°;$$

$$\mu_d = 0 - 252° = -252°.$$

Substituting in (10), respectively, $\nu = 1$ and the values μ obtained, and also carrying out an approximate integration according to the Chebyshev formula [5], we obtain

$$\eta_1 = 0.332, \quad \eta_2 = 0.0075,$$

$$\eta_d = 0.00268.$$

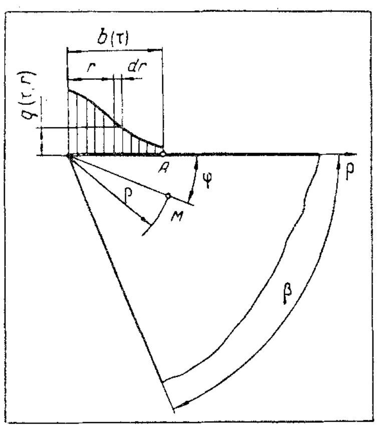

Fig. 1. Schematic location of heat source on the boundary surface of an infinite wedge.

Then the dimensionless temperature at point A will be

$$T = 2\sum_{n=1}^{k} \eta_n + \delta\eta_d = 2(\eta_1 + \eta_2) + 1 \cdot \eta_d =$$

$$= 2(0.332 + 0.0075) + 0.00268 = 0.683.$$

The temperatures at other points may be calculated similarly. Figure 2 shows the temperature distribution on the boundaries of the wedge, calculated for the example examined.

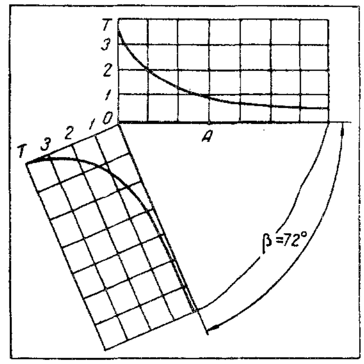

Fig. 2. Distribution of dimensionless temperature on the boundary planes of the wedge.

The solution derived may be used to calculate the temperature in cutting tools during intermittent cutting processes (milling, discontinuous turning, etc.).

Notation:

β—wedge angle; τ—time of action of instantaneous source element; q—intensity of heat source; r—polar coordinate; σ—characteristic of normal distribution curve; q_l—intensity of linear source; λ—thermal conductivity; a—thermal diffusivity; R—distance from source element to point examined; t—time; ρ and φ—polar coordinates of point examined; τ_0—duration of heating; μ—angle between source and point.

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Kuibyshev Polytechnic Institute, Kuibyshev